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CALCULATIONS OF THE EQUATIONS OF A THREE-DIMENSIONAL LAMINAR BOUNDARY LAYER BY THE INTEGRAL METHOD

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The integral method proposed by A. A. Dorodnitsyn [1] for solving nonlinear aero- /1280* dynamical problems is used extensively for numerical integration of laminar boundary layer equations.

Integration of the equations of a planar or axisymmetric laminar boundary layer [2-6] by this method has shown that as the number of division strips of the field of flow increased, the difference between two consecutive approximations decrease and the solution obtained is in good agreement with calculations by other methods. This indicates that as the number of strips is increased the accuracy of the calculation also increases and the solution itself is a solution of the original differential equation. In [3-5], an estimate was made of the accuracy of calculation of the equations of a laminar boundary layer by the integral method from the values of local frictional stresses and local heat flux. In particular, it is shown in [3] that, in a neighborhood of a planar stagnation point, the local frictional stress and the local heat flux are calculated in the fourth approximation with an error of the order of 0.2%. Calculation of a laminar boundary layer on a blunted (rounded) plate [4] showed that, for $\bar{x} \leq \pi/2$, the fourth and third approximations differ by less than 3%, whereas for $x > \pi/2$, the convergence is worse and the difference amounts to about 6%. The fourth and fifth approximations in cylindrical blunting differ by less than 1.5% and outside it, the difference amounts to about 3% [5].

Comparison of the results of calculation of the characteristics of a laminar axisymmetric boundary layer by the integral method with the data of numerical integration [6] has shown that, when the field of flow is divided into five strips, the method of integral relationships in a region with zero or negative pressure gradient ensures computational accuracy of the order of 1.5% in the case of high heat transfer rates and of the order of 1.5-20% without heat transfer. In a region with a positive pressure gradient, the results of the calculations by these two methods differ considerably from one another.

In the present article, the method of integral relationships is applied to a numerical /1281 integration of the equations of three-dimensional laminar boundary layer when there is conical external flow. With three particular examples an estimate is made of the accuracy of calculation of the local friction factor and of the local heat flux and also of the profiles of the frictional stress, the rate of secondary flow, and the total enthalpy.

The method of integral relationships was applied earlier by the author to integrations of equations of a laminar boundary layer on infinite elliptic sliding cylinders [7]. However, in that article, no estimate was made for the accuracy of the calculation.

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*Numbers in the margin indicate pagination in the foreign text.

1.. The system of differential equations of three-dimensional laminar boundary layer has the form [8]

$$\begin{aligned}
&\frac{\partial}{\partial\xi}(h_2\rho u)+\frac{\partial}{\partial\eta}(h_1\rho v)+\frac{\partial}{\partial\zeta}(h_1h_2\rho w)=0,\\
&\frac{u}{h_1}\frac{\partial u}{\partial\xi}+\frac{v}{h_2}\frac{\partial u}{\partial\eta}+w\frac{\partial u}{\partial\zeta}+\frac{uv}{h_1h_2}\frac{\partial h_1}{\partial\eta}-\frac{v^2}{h_1h_2}\frac{\partial h_2}{\partial\xi}=-\frac{1}{\rho h_1}\frac{\partial p}{\partial\xi}+\frac{1}{\rho}\frac{\partial}{\partial\zeta}\left(\mu\frac{\partial u}{\partial\zeta}\right),\\
&\frac{u}{h_1}\frac{\partial v}{\partial\xi}+\frac{v}{h_2}\frac{\partial v}{\partial\eta}+w\frac{\partial v}{\partial\zeta}+\frac{uv}{h_1h_2}\frac{\partial h_2}{\partial\xi}-\frac{u^2}{h_1h_2}\frac{\partial h_1}{\partial\eta}=-\frac{1}{\rho h_2}\frac{\partial p}{\partial\eta}+\frac{1}{\rho}\frac{\partial}{\partial\zeta}\left(\mu\frac{\partial v}{\partial\zeta}\right),\\
&\frac{u}{h_1}\frac{\partial H}{\partial\xi}+\frac{v}{h_2}\frac{\partial H}{\partial\eta}+w\frac{\partial H}{\partial\zeta}=\frac{1}{\rho}\frac{\partial}{\partial\zeta}\left\{\mu\left[\frac{1}{P}\frac{\partial H}{\partial\zeta}-\frac{1-P}{P}\frac{\partial}{\partial\zeta}\left(\frac{u^2+v^2}{2}\right)\right]\right\}.
\end{aligned}\tag{1.1}$$

The solution of this system of equations when a homogeneous gas is either blown in or sucked out must satisfy the boundary conditions

$$\begin{aligned}
&u=v=0, \quad w=w_w, \quad H=H_w, \quad \zeta=0;\\
&u\to u_e, \quad v\to v_e, \quad H\to H_e, \quad \zeta\to\infty.
\end{aligned}$$

Here, ξ, η, ζ constitute an arbitrary orthogonal coordinate system and the surface $\zeta=0$ coincides with flow around the surface; $h_1 = h_1(\xi, \eta)$ and $h_2 = h_2(\xi, \eta)$ are the Lame coefficients, u, v, and w are the components of the velocity vector parallel to the coordinate axes ξ, η and ζ respectively; o, p, μ, and H denote the density, pressure, coefficient of dynamic viscosity, and the total enthalpy (heat content) of the gas respectively; Prandtl number, and the subscript e refer to the flow variables at the external boundary of the boundary layer, and the subscript w refers to the surface of the body.

Consider the special case when the flow of a compressed gas at the external boundary of the boundary layer is conical. We take for coordinates on the surface around which the flow is taking place the coordinates r, 0, where r denotes the coordinate along a ray issuing from the vertex of the body and lying on its surface and 0 denotes the polar angle in the case of a planar triangular wing or the angle characterizing the position of the meridional plane orthogonal to the surface of the boundary (circular and elliptic cones).

For conical bodies, the Lame coefficients have the forms

$$h_1=1, \quad h_2=r\psi(\theta),$$

where $\psi(\theta)$ is a function whose form is determined by the geometry of the body. The assumption that the flow is conical imposes a definite relationship between the parameters of a nonviscous flow, which follows from the differential flow equations /1282

$$\frac{v_e}{u_e^2\psi}\frac{du_e}{d\theta}-\left(\frac{v_e}{u_e}\right)^2=0.$$

We shall consider the flow of a gas in a laminar boundary layer in Crocco's variables r, 0, u. To solve the problem we introduce the dimensionless quantities

$$u_1 = \frac{u}{u_e}, \quad \beta_2 = \frac{v}{v_e}, \quad \beta_3 = \frac{\partial \beta_2}{\partial u_1}, \quad \beta_4 = \frac{H}{H_e},$$

$$\beta_5 = \frac{\partial \beta_4}{\partial u_1}, \quad \beta_6 = \frac{\rho\mu}{\rho_e\mu_e}, \quad \rho_1 = \frac{\rho}{\rho_e},$$

$$\beta_0 = \frac{\tau_1 \sqrt{R_2}}{\rho_e u_e^2 \beta_6 \sqrt{(c+\alpha_1)}}, \quad \beta_1 = \frac{\partial \beta_0}{\partial u_1}, \quad R_r = \frac{\rho_e u_e r}{\mu_e},$$

$$\alpha_0 = \frac{u_e^2}{2H_e}, \quad \alpha_1 = \frac{1}{\psi u_e}\frac{dv_e}{d\theta}, \quad \alpha_2 = \frac{v_e}{u_e^2 \psi}\frac{du_e}{d\theta} = \left(\frac{v_e}{u_e}\right)^2, \quad \alpha_3 = \frac{v_e}{\psi u_e}\frac{d \ln \rho_e \mu_e}{d\theta},$$

$$\alpha_4 = \frac{v_e}{u_e \psi}\frac{d\psi}{d\theta}, \quad \alpha_5 = \frac{v_e}{\psi u_e}, \quad \alpha_6 = \frac{v_e}{u_e^2 \psi^2}\frac{d^2 v_e}{d\theta^2}, \quad \alpha_7 = \frac{\rho_w w_w}{\rho_e u_e}\sqrt{\left(\frac{R_r}{c+\alpha_1}\right)}.$$

Here, $\tau_1 = \mu \partial u / \partial \zeta$ is the longitudinal component of the frictional stress, c is a constant ≥ 1. Then, the system of differential equations (1.1) becomes

$$\frac{\alpha_5}{c+\alpha_1}\frac{\partial}{\partial \theta}\left(\frac{\beta_2}{\beta_0}\right) = -\left\{\frac{\partial^2(\beta_0\beta_6)}{\partial u_1^2} + \frac{1.5u_1}{(c+\alpha_1)\beta_0} + \frac{\alpha_1\beta_2}{(c+\alpha_1)\beta_0}\right\} -$$
$$-\left\{\frac{\alpha_2}{c+\alpha_1}\left[\frac{\beta_2\beta_1(u_1-\beta_2)}{\beta_0^2} + \frac{(2\beta_2-u_1)\beta_3}{\beta_0}\right] + \right.$$
$$\left. + \left[\frac{\alpha_3 - 3\alpha_2}{2(c+\alpha_1)} + \frac{\alpha_1(\alpha_2+\alpha_4)-\alpha_6}{2(c+\alpha_1)^2}\right]\frac{\beta_2}{\beta_0}\right\},$$

$$\frac{\alpha_5\beta_2}{(c+\alpha_1)\beta_6\beta_0^2}\frac{\partial \beta_2}{\partial \theta} = \frac{\partial^2 \beta_2}{\partial u_1^2} + \frac{\alpha_2\beta_2\beta_3(u_1-\beta_2)}{(c+\alpha_1)\beta_6\beta_0^2} + \tag{1.2}$$
$$+ \frac{(1+\alpha_1)\rho_1^{-1} - \alpha_1\beta_2^2 - u_1\beta_2}{(c+\alpha_1)\beta_6\beta_0^2},$$

$$\frac{\alpha_5\beta_2}{(c+\alpha_1)\beta_6\beta_0^2}\frac{\partial \beta_4}{\partial \theta} = \left\{\frac{\partial}{\partial u_1}\left(\frac{\beta_5}{P}\right) + \frac{1-P}{P}\frac{\partial \ln \beta_0\beta_6}{\partial u_1}\beta_5 - \right.$$
$$\left. - \frac{2\alpha_0}{\beta_6\beta_0}\frac{\partial}{\partial u_1}\left(\frac{1-P}{P}u_1\beta_6\beta_0\right)\right\} + \left\{\frac{\alpha_2(u_1-\beta_2)\beta_2\beta_5}{(c+\alpha_1)\beta_6\beta_0^2} - \right.$$
$$\left. - \frac{2\alpha_0\alpha_2}{\beta_6\beta_0}\frac{\partial}{\partial u_1}\left(\frac{1-P}{P}\beta_6\beta_0\beta_2\beta_3\right)\right\}$$

with boundary conditions

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$$\frac{\partial(\beta_6\beta_0)}{\partial u_1} = \alpha_7, \quad \beta_2 = 0, \quad \beta_4 = H_{1w}, \quad u_1 = 0,$$
$$\beta_0 = 0, \quad \beta_2 = \beta_4 = 1, \quad u_1 = 1.$$

Since all the α_i must be functions of θ only, the local flow rate of a homogenous gas must be inversely proportional to the square root of the longitudinal coordinate r.

The differential equation for determining the displacement thickness δ^* in the case of a three-dimensional boundary layer is obtained from the flow rate balance equation through an arbitrary closed loop [9]. In the notation introduced above, we have

$$\frac{a_5}{c+a_1}\frac{dT}{d\theta}+\frac{1}{2}\left\{3+2a_1+a_3-a_2+\frac{a_1(a_4+a_2)-a_6}{c+a_1}\right\}\frac{T}{c+a_1}= \\ =a_7-\frac{3}{2}\frac{\delta_1-\delta_2}{c+a_1}, \tag{1.3}$$

where

$$\delta_1=\int_0^1\frac{1-\rho_1\beta_2}{\rho_1\beta_0}du_1, \quad \delta_2=\int_0^1\frac{1-\rho_1 u_1}{\rho_1\beta_0}du_1,$$

$$T(\theta)=\chi(0)-\delta_1, \quad \chi(0)=\frac{\delta^*}{r}\sqrt{[R_r(c+a_1)]}.$$

2. For a numerical integration of the system of equations (1.2), we apply the method of integral relationships proposed by A. A. Dorodnitsyn [2]. We shall approximate the profiles of the functions with polynomials that satisfy the boundary conditions on the outer boundary of the boundary layer:

$$\beta_0\beta_6=(1-u_1)\sum_{k=0}^{m}a_{0k}u_1^k, \quad \left.\begin{matrix}\beta_2\\ \beta_4\end{matrix}\right\}=u_1+(1-u_1)\sum_{k=0}^{m}a_{2k}u_1^k.$$

The field of flow is divided into m + 1 strips and the coefficients a_{0k} and a_{2k} and are expressed in terms of the values assumed on the boundaries of these strips by the functions to be approximated.

Let us group the terms in the differential equations (1.2) in such a way that on the external boundary of the boundary layer they have a singularity in the form of a first-order pole. Then, the system of functions $\varphi_n(u_1)=(1-u_1)^n$, where n = 1, 2, 3, ..., m will serve as a system of orthogonalizing functions for all the equations.

If we multiply the differential equations (1.2) by $\varphi_n(u_1)$ and integrate across the boundary layer, these equations become the system of ordinary differential equations

$$\sum_{i=1}^{m}\frac{a_5C_n^{(i)}(0)\beta_2^{(i)}}{(c+a_1)\beta_0^{(i)2}}\frac{\partial\beta_0^{(i)}}{\partial\theta}=\sum_{i=1}^{m}\frac{a_5C_n^{(i)}(0)}{(c+a_1)\beta_0^{(i)}}\frac{\partial\beta_2^{(i)}}{\partial\theta}+\Psi_{1,n}^{(1)}(\theta)+\Psi_{1,n}^{(2)}(\theta), \tag{/1284}$$

$$\sum_{i=1}^{m}\frac{a_5C_n^{(i)}(0)\beta_2^{(i)}}{(c+a_1)\beta_6^{(i)}\beta_0^{(i)2}}\frac{\partial\beta_2^{(i)}}{\partial\theta}=\Psi_{2,n}^{(1)}(\theta)+\Psi_{2,n}^{(2)}(\theta),$$
$$\sum_{i=1}^{m}\frac{a_5C_n^{(i)}(0)\beta_2^{(i)}}{(c+a_1)\beta_6^{(i)}\beta_0^{(i)2}}\frac{\partial\beta_4^{(i)}}{\partial\theta}=\Psi_{3,n}^{(1)}(\theta)+\Psi_{3,n}^{(2)}(\theta). \tag{2.1}$$

Here, we use the notation

$$\Psi^{(1)}_{1,n}(\theta) = -\alpha_7 - n\beta_0^{(0)}\beta_6^{(0)} + \sum_{i=0}^{m}\left\{nA^{(i)}_{n-1}(0)\,\beta_6^{(i)}\beta_0^{(i)} + \right.$$
$$\left. + \frac{1.5C_n^{(i)}(1)}{(c+\alpha_1)\,\beta_0^{(i)}} + \frac{\alpha_1 C_n^{(i)}(0)\,\beta_2^{(i)}}{(c+\alpha_1)\,\beta_0^{(i)}}\right\},$$

$$\Psi^{(2)}_{1,n}(\theta) = \sum_{i=0}^{m} C_n^{(i)}(0)\left\{\frac{\alpha_2}{c+\alpha_1}\left[\frac{\beta_2^{(i)}\beta_1^{(i)}}{\beta_0^{(i)2}}\left(\frac{i}{m+1} - \beta_2^{(i)}\right) + \right.\right.$$
$$\left.\left. + \frac{\beta_3^{(i)}}{\beta_0^{(i)}}\left(2\beta_2^{(i)} - \frac{i}{m+1}\right)\right] + \left[\frac{\alpha_3 - 3\alpha_2}{2(c+\alpha_1)} + \frac{\alpha_1(\alpha_2+\alpha_4) - \alpha_6}{2(c+\alpha_1)^2}\right]\frac{\beta_2^{(i)}}{\beta_0^{(i)}}\right\},$$

$$\Psi^{(1)}_{2,n}(\theta) = -\beta_3^{(0)} + n\widetilde{A}_{n-1}(1.0) + n\sum_{i=0}^{m} A^{(i)}_{n-1}(0)\,\beta_2^{(i)} +$$
$$+ \sum_{i=0}^{m}\frac{C_n^{(i)}(0)}{(c+\alpha_1)\,\beta_6^{(i)}\beta_0^{(i)2}}\left\{\frac{1+\alpha_1}{\rho_1^{(i)}} - \alpha_1\beta_2^{(i)2} - \frac{i\beta_2^{(i)}}{m+1}\right\},$$

$$\Psi^{(2)}_{2,n}(\theta) = \sum_{i=0}^{m}\frac{\alpha_2 C_n^{(i)}(0)\,\beta_2^{(i)}\beta_3^{(i)}}{(c+\alpha_1)\,\beta_6^{(i)}\beta_0^{(i)2}}\left(\frac{i}{m+1} - \beta_2^{(i)}\right),$$

$$\Psi^{(1)}_{3,n}(0) = -\frac{\beta_5^{(0)}}{p^{(0)}} - 2\alpha_0\frac{1-p^{(m+1)}}{p^{(m+1)}}\,\widetilde{A}_n(1.1) + \sum_{i=0}^{m}\left\{nD^{(i)}_{n-1}(0)\,\frac{\beta_5^{(i)}}{p^{(i)}} + \right.$$
$$+ \frac{1-p^{(i)}}{p^{(i)}}\left[C_n^{(i)}(0)\left(\frac{\partial \ln \beta_6\beta_0}{\partial u_1}\right)^{(i)}\beta_5^{(i)} - 2\alpha_0 A_n^{(i)}(1) - \right.$$
$$\left.\left. - 2\alpha_0 C_n^{(i)}(1)\left(\frac{\partial \ln \beta_6\beta_0}{\partial u_1}\right)^{(i)}\right]\right\},$$

$$\Psi^{(2)}_{3,n}(0) = \sum_{i=0}^{m}\left\{\frac{\alpha_2 C_n^{(i)}(0)\,\beta_2^{(i)}\beta_5^{(i)}}{(c+\alpha_1)\,\beta_6^{(i)}\beta_0^{(i)2}}\left(\frac{i}{m+1} - \beta_2^{(i)}\right) - \right.$$
$$\left. - 2\alpha_0\alpha_2\beta_2^{(i)}\beta_3^{(i)}\frac{1-p^{(i)}}{p^{(i)}}\left[nD^{(i)}_{n-1}(0) + C_n^{(i)}(0)\left(\frac{\partial \ln \beta_6\beta_0}{\partial u_1}\right)^{(i)}\right]\right\};$$

$\widetilde{A}_{n-1}(1.0)$, $\widetilde{A}_n(1.1)$, etc., are constant coefficients depending on n and the total number of strips.

The system of equations (2.1) is incomplete but can be completed by relations following from the boundary conditions on the surface of the body.

If the solution of the system of equations (2.1) has been found, then the local frictional factors and the local heat flux are calculated according to the formulas

$$c_{f1}\sqrt{R_{r\infty}} = \frac{\tau_{1w}\sqrt{R_{r\infty}}}{0.5\rho_\infty u_\infty^2} = 2\frac{u_e}{u_\infty}\sqrt{\left[\frac{\rho_e\mu_e u_e}{\rho_\infty\mu_\infty u_\infty}(c+\alpha_1)\right]\beta_6^{(0)}\beta_0^{(0)}},$$

$$c_{f2}\sqrt{R_{2\infty}} = \frac{\tau_{2w}\sqrt{R_{r\infty}}}{0.5\rho_\infty u_\infty^2} = \sqrt{\alpha_2}\,c_{f1}\sqrt{R_{r\infty}\,\beta_3^{(0)}},$$

$$c_{f\Sigma}\sqrt{R_{r\infty}} = \sqrt{[(c_{f1}^2 + c_{f2}^2)\,R_{r\infty}]},$$

$$\bar{q}_w\sqrt{R_{r\infty}} = \frac{q_w\sqrt{R_{r\infty}}}{\rho_\infty u_\infty H_\infty} = \frac{1}{2}\frac{u_\infty}{u_e}c_{f1}\sqrt{R_{r\infty}}\frac{\beta_5^{(0)}}{p^{(0)}}.$$

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Here, τ_{1w} and τ_{2w} are the longitudinal and transverse components of the frictional stress and q_w is the local heat flux.

3. For numerical integration of the system of equations (2.1) and equation (1.3), a program has been set up for an arbitrary value of the parameter $m \leq 12$ for use in the electronic computer M-20. Integration of the equations of the boundary layer begins with the streamline ($\alpha_i = 0$, for $i = 2, 3, \ldots, 6$).

On a streamline, the solution of the problem reduces to numerical integration of a system of nonlinear algebraic equations by the method of successive approximations with a given degree of accuracy.

The finding of the solution of the system of ordinary differential equations (2.1) and equation (1.3) is a Cauchy problem with initial conditions that can be determined as a consequence of the solution of the equations of a boundary layer on a streamline. Numerical integration of this system of equations is done by the Runge-Kutta method in accordance with second-order formulas. To depart from a streamline we are given the initial step of integration and the zero values of the derivatives

$$d\beta_0^{(i)}/d\theta,\ d\beta_2^{(i)}/d\theta,\ d\beta_4^{(i)}/d\theta$$

on it by virtue of the evenness of the functions β_0, β_2, and β_4 in a neighborhood of a streamline. Then, the calculation is carried out automatically with a variable step of integration.

Control calculations have shown that, to avoid accumulation of error, the calculation should be made with step $\Delta\theta = \alpha_5/K(c + \alpha_1)$. For $c = 2$ and various values of the parameter m, the value of the constant K was chosen as follows:

m	3	4	5	6
K	50	100	200	350

On the basis of these data, the following was established for calculating the step of integration:

$$\Delta\theta = \frac{\alpha_5}{(c + \alpha_1)\,50 \cdot 2^{m-3}}.$$

It was used in the program for calculating the step of integration for $c \geq 2$.

With the calculating scheme that was chosen, increase in the total number of strips leads to decrease in the accuracy of the calculation on the part of the machine. This shows up sharply in the solution of the approximating system of equations on a streamline. For $m \leq 6$, the process of successive approximations converges with absolute accuracy $\varepsilon = 10^{-8}$, with $\varepsilon = 10^{-7}$ for $m = 7$, with $\varepsilon = 4 \cdot 10^{-6}$ with $m = 8$. This decrease in the accuracy of calculation on the part of the machine is connected with the fact that the

approximating system of equations becomes more poorly conditioned with increase in its order and the discharge net of the machine is restricted and it determines the upper value of the number of strips for which the accuracy of calculation of the equations of the boundary layer coincides with the machine calculating accuracy of the approximating system of equations on a streamline. To carry out the calculations with a greater number of strips, it is necessary to change the calculating scheme.

An estimate of the accuracy of calculation of the equations of a laminar boundary layer was made for the three special cases:

1) The streamline $\alpha_0 = 0$, $\alpha_1 = 0$;

2) the streamline $\alpha_0 = 0$, $\alpha_1 = 1$;

3) an acute circular cone with half-angle at the vertex $0_k = 15°$ with an angle of incidence $\alpha = 10°$ and with $M_\infty = 7$.

Functions	the Runge Kutta method	$m=3$	4	5	6	7	8
			$\alpha_0 = 0,\ \alpha_1 = 0,\ H_{1w} = 0.05$				
$\beta_0^{(0)}\beta_6^{(0)}$	0.45000	0.43714	0.46016	0.44585	0.45322	0.44832	0.45106
$\varepsilon\%$		—2.86	2.26	—0.92	0.71	—0.37	0.24
$\beta_3^{(0)}$	1.28826	1.30934	1.26837	1.29142	1.28121	1.28920	1.28570
$\varepsilon\%$		1.64	—1.54	0.24	—0.55	0.08	—0.20
$\beta_5^{(0)}$	0.83356	0.81906	0.83347	0.82818	0.83360	0.83213	0.83487
$\varepsilon\%$		—1.75	—0.02	—0.66	—0.01	—0.18	0.14
$\chi^{(0)}$	—0.0992	—0.1382	—0.0938	—0.1213	—0.1031	—0.1114	—0.1011
$\varepsilon\%$		39.30	—6.45	22.30	3.93	12.30	1.92
			$\alpha_0 = 0,\ \alpha_1 = 1,\ H_{1w} = 0.05$				
$\beta_0^{(0)}\beta_6^{(0)}$	0.48899	0.48294	0.49344	0.48666	0.49045	0.48801	0.48944
$\varepsilon\%$		—1.24	0.91	—0.48	0.30	—0.02	0.01
$\beta_3^{(0)}$	1.29775	1.30667	1.28484	1.29758	1.29296	1.29711	1.29560
$\varepsilon\%$		0.69	—1.00	—0.01	—0.37	—0.05	—0.16
$\beta_5^{(0)}$	0.83173	0.82379	0.83025	0.82809	0.83135	0.83100	0.83286
$\varepsilon\%$		—0.95	—0.18	—0.44	—0.04	—0.09	0.14
$\chi^{(0)}$	—0.1564	—0.1935	—0.1522	—0.1765	—0.1620	—0.1687	—0.1607
$\varepsilon\%$		23.70	—2.69	6.46	3.58	7.87	2.75
			$M_\infty = 7,\ 0_k = 15°,\ \alpha = 10°,\ 0 = 0,\ H_{1w} = 0.05$				
$c_{f1}\sqrt{R_{r\infty}}$		3.73756	3.81210	3.76291	3.78904	3.77198	3.78128
$\Delta_{m+1,\,m}$		—	0.07454	—0.04950	0.02644	—0.01707	0.00930
$\bar{q}_w\sqrt{R_{r\infty}}$		2.21182	2.26627	2.22768	2.24675	2.23243	2.23949
$\Delta_{m+1,\,m}$		—	0.05445	—0.03860	0.01907	—0.01432	0.00707
$\chi^{(0)}$		1.04720	1.03656	1.04528	1.03949	1.03695	1.04286
$\Delta_{m+1,\,m}$		—	—0.01064	0.00872	—0.00578	—0.00254	0.00591

The calculations were made under the assumption of a perfect gas (P = 0.7, $\mu \sim T^{0.76}$), and of considerable heat transfer ($H_{1w} = 0.05$) on the impenetrable surface of the body.

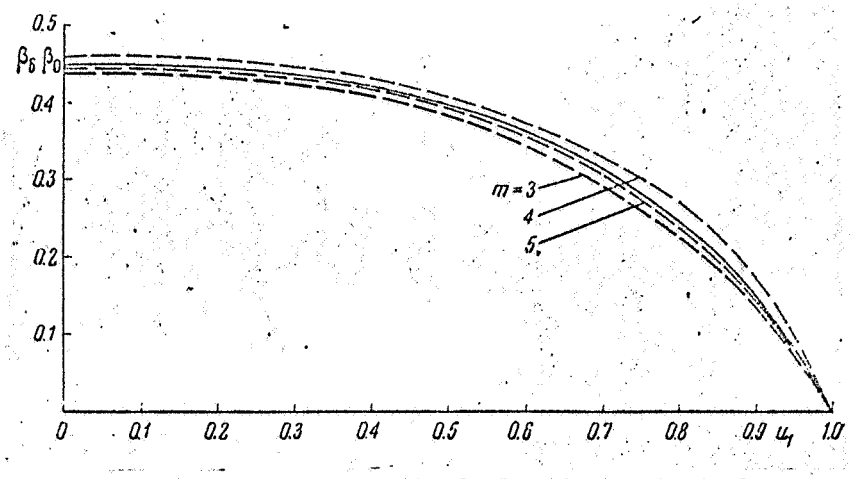

Fig. 1. The Runge-Kutta Method: The Method of Integral Relationships: $\alpha_0 = 0$, $\alpha_1 = 0$.

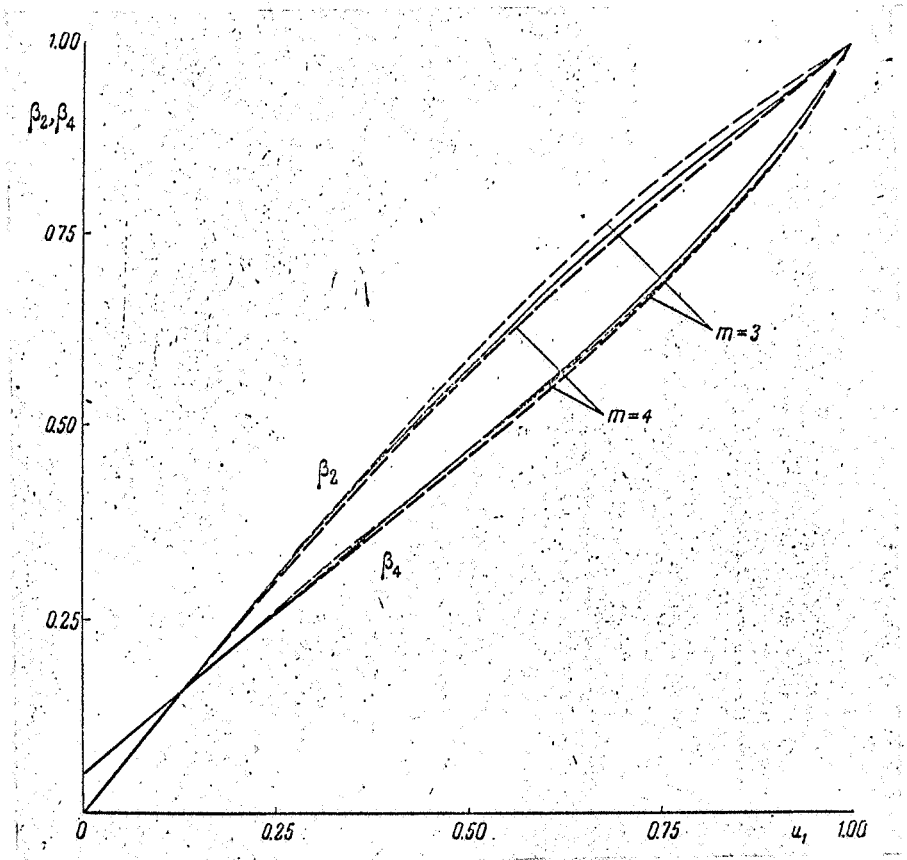

Fig. 2. The Runge-Kutta Method: The Method of Integral Relationships: $\alpha_0 = 0$, $\alpha_1 = 0$.

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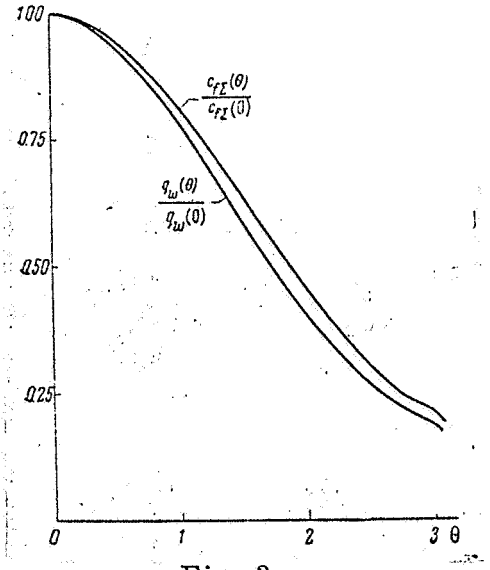

Fig. 3.

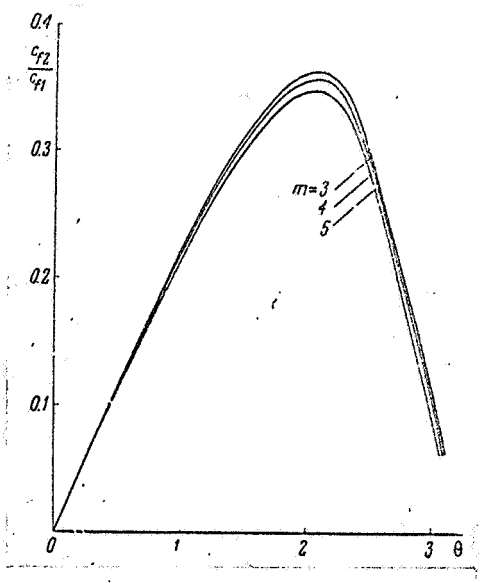

Fig. 4.

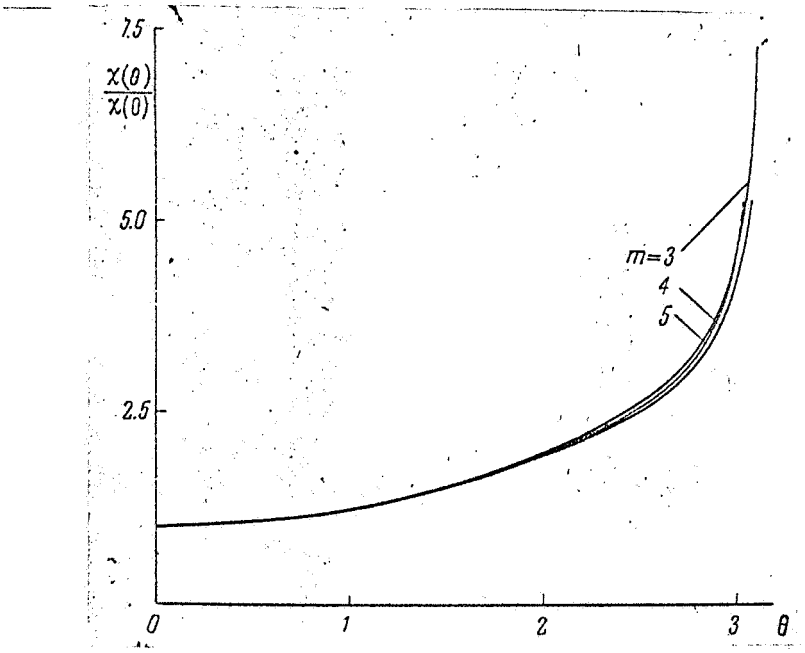

Fig. 5.

The solution of the equations of a laminar boundary layer on the streamlines is self-similar. Therefore, for the first two cases we used as an exact solution the solution obtained in [10] by the method of numerical integration of ordinary differential equations by the Runge-Kutta method. For an acute circular cone with angles of incidence $\alpha < 0_k$, certain results of the calculations of the equations of a boundary layer /1289 that were obtained by the method of finite differences are given in [11]. However, for the case that we are considering, the results of the calculations are absent. Therefore, in the third case, an investigation was made only of the process of convergence of the results of the calculations as a function of the total number of strips.

The results of calculations of the quantities $\beta_6^{(0)}\beta_0^{(0)}$, $\beta_3^{(0)}$, $\beta_5^{(0)}$ on a streamline for the first two cases are given in the Table.

For a fixed value of the parameter m, the maximum computational error, that is, the deviation from the results of calculations made according to the Runge-Kutta method, occurs in the calculations of $\beta_u^{(\omega)}\beta_0^{(\omega)}$. For m = 3, it is less than 3%. As we increase m, the error in the calculation decreases in absolute value and it is a sign-variable function. As a function of m, it obeys approximately the exponential law

$$\varepsilon = (-1)^m |\varepsilon_{\max}| \exp\{-0.51(m-3)\}.$$

Thus, for m = 3, the local frictional factors and the local heat flux are calculated with a degree of accuracy that is acceptable in practice ($\varepsilon < 3\%$).

Figures 1 and 2 show the results of calculations of the profiles of the frictional stress, the rate of secondary flow, and the total enthalpy on the streamlines. One can see, that with a fixed number of strips in the partition, the approximation error is minimum close to the wall and maximum in a neighborhood of the outer boundary of the boundary layer. To obtain acceptable accuracy in the calculation throughout the entire field of flow, one needs to choose a sufficiently large value for the parameter m.

Such nonuniform converge with respect to m at different points of the field of flow causes a large error in calculation of the displacement thickness. For example, for m = 3, it amounts to about 30%, whereas the local friction factors and the local heat flux are calculated with an accuracy of the order of 3%. An acceptable accuracy in the calculation of the displacement thickness is attained when we have a sufficiently large number of strips (see Table).

Figures 3-5 show the distribution of the friction factors the heat flux, and the displacement thickness in a cross section of the cone in the form of the dependence

$$c_{f2}(0)/c_{f1}(0),\quad c_{f\Sigma}(0)/c_{f\Sigma}(0),\quad q_w(0)/q_w(0) \text{ и } \chi(0)/\chi(0)$$

on the angle θ for m = 3, 4, 5. One can see that the distribution of these quantities in a cross section of the cone depends in a comparatively weak way on the number of strips. The maximum difference between the two approximations corresponding to m = 3 and m = 4 for $c_{f2}(\theta)/c_{f1}(\theta)$ is of the order of 4%; for $c_{f\Sigma}(\theta)/c_{f\Sigma}(\theta)$ and $q_w(\theta)/q_w(\theta)$, it is of the order of 1.5%; for $\chi(\theta)/\chi(0)$, it is of the order of 9%. /1290

Thus, the results of the calculations of a laminar boundary layer by the method of integral relationships for three special cases show that, when the field of flow is divided into four strips (m = 3) this method enables us to calculate the values of the local frictional stress and the local heat flux with an acceptable degree of accuracy ($\varepsilon < 3\%$). However, in this case, the displacement thickness is calculated with a very large error ($\varepsilon \leqslant 30$-40%). To calculate it with an acceptable degree of accuracy ($\varepsilon \leqslant 3\%$), it is necessary to carry out the calculations with a large number of strips ($m \leq 8$).

Weak dependence of the ratio $\chi(\theta)/\chi(\theta)$ on the parameter m enables us to carry out all the calculations for a fixed value of m (let us say, for m = 3), and then carry out the calculations on a streamline for a larger number of strips (for example, with m = 8)

and calculate the initial parameters of the boundary layer with a considerably higher degree of accuracy. /1290

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